Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

Journal of Combinatorial Theory, Series A 105 (2004) 63–77

Journal of Combinatorial Theory
Series A

http://www.elsevier.com/locate/jcta

Combinatorial proofs of Ramanujan's ${}_1\psi_1$ summation and the q-Gauss summation

Ae Ja Yee[1]

Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

Received 4 March 2003

Abstract

Theorems in the theory of partitions are closely related to basic hypergeometric series. Some identities arising in basic hypergeometric series can be interpreted in the theory of partitions using F-partitions. In this paper, Ramanujan's ${}_1\psi_1$ summation and the q-Gauss summation are established combinatorially.

1. Introduction

Frobenius [12] studied two-rowed arrays of distinct integers in decreasing order in his work on group representation theory. His idea has been employed primarily in the representation theory of symmetric groups. In 1984, Andrews [4] generalized the idea to the theory of partitions giving various restrictions on integers in two-rowed arrays. He called two-rowed arrays generalized Frobenius partitions or simply F-partitions. Some identities arising in q-series can be interpreted in the theory of partitions using F-partitions. In this paper, Ramanujan's ${}_1\psi_1$ summation and the q-Gauss summation are combinatorially proved.

In 1987, Joichi and Stanton [19] established bijective proofs of basic hypergeometric series identities by systematical methods. Their strategy is to set up an involution for each step and to form a bijection for an identity by combining the involutions. In their paper, central is the q-binomial theorem

$$\prod_{i=1}^{\infty} \frac{1+q^i}{1-q^i} = 1 + \sum_{m=1}^{\infty} \frac{(1+1)(1+q)\cdots(1+q^{m-1})}{(1-q)(1-q^2)\cdots(1-q^m)} q^m \quad \text{for } |q|<1, \tag{1.1}$$

E-mail address: yee@math.psu.edu.

[1] Research partially supported by a grant from the Number Theory Foundation.

doi:10.1016/j.jcta.2003.10.002

which is the generating function of overpartitions introduced recently by Corteel and Lovejoy [10], where an overpartition of n is a partition of n where the first occurrence of a number may be overlined. Besides the combinatorial proof of Joichi and Stanton, another combinatorial proof of (1.1) is found in [21]. We describe another bijective proof of (1.1), which is a variation of the bijection of Joichi and Stanton as well as a starting point of this work. We discuss this new bijection in Section 2.

In the sequel, we assume that $|q|<1$ and use the customary notation for q-series

$$\begin{aligned}(a)_0 &:= (a;q)_0 := 1,\\ (a)_\infty &:= (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),\\ (a)_n &:= (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty} \quad \text{for any } n.\end{aligned}$$

The well-known Ramanujan's ${}_1\psi_1$ summation is given by

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n = \frac{(b/a;q)_\infty(q;q)_\infty(az;q)_\infty(q/az;q)_\infty}{(b;q)_\infty(q/a;q)_\infty(b/az;q)_\infty(z;q)_\infty} \quad \text{for } |b/a|<|z|<1. \tag{1.2}$$

In the theory of q-series, the ${}_1\psi_1$ summation is a central theorem. Hardy [15, pp. 222, 223] described it as a remarkable formula with many parameters. The first published proofs appear to be by Hahn [14] and Jackson [18] in 1949 and 1950, respectively. Jackson and her advisor Bailey [7] were the first to write it in the form (1.2). Other proofs have been given by Adiga et al. [1], Andrews [2,3], Andrews and Askey [5], Askey [6], Fine [11], Ismail [17], and Mimachi [20].

Recently, Corteel and Lovejoy [9] have given a combinatorial proof of the constant term, and Corteel [8] has completed their combinatorial proof using particle seas. However, her proof is not a bijection between two sets arising from both sides of the ${}_1\psi_1$ summation. In Section 3, we establish a natural combinatorial proof. In fact, we give a second bijective proof, which is described in Section 5.

In the theory of basic hypergeometric series, the q-Gauss summation plays an important role. The q-Gauss summation [13] is given by

$$\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}(c/ab)^n = \frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty} \quad \text{for } |c/ab|<1, \tag{1.3}$$

which was first proved by Heine [16]. The q-Gauss summation is normally proved by Heine's transformation formula. However, in this paper, we see that the q-Gauss summation is related to Ramanujan's ${}_1\psi_1$ summation as mentioned and proved in [8,9]. In Section 4, we explain the q-Gauss summation in a combinatorial way using the bijection for the ${}_1\psi_1$ summation.

2. F-Partitions

For a nonnegative integer n, a Frobenius partition of n is a two-rowed array of row-wise distinct nonnegative integers

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where each row is of the same length, each is arranged in decreasing order, and $n = r + \sum_{i=1}^{r}(a_i + b_i)$. A Frobenius partition is another representation of an ordinary partition. Putting r dots in the main diagonal direction and attaching a_i dots to the right of the diagonal in rows and b_i dots below the diagonal in columns, we can obtain a Ferrers graph of an ordinary partition. For example, the Frobenius partition

$$\begin{pmatrix} 6 & 5 & 2 & 0 \\ 5 & 4 & 1 & 0 \end{pmatrix}$$

corresponds to the partition $7+7+5+4+2+2$, as seen easily from the Ferrers graph in Fig. 1. In particular, Frobenius partitions can be used to prove Jacobi's triple product identity [4,22,23]. Andrews has generalized Frobenius partitions in [4].

For sets A and B of certain kinds of partitions, we define

$$\mathscr{F}(A,B) = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_s \end{pmatrix} \middle| (a_1\ a_2\ \cdots\ a_r) \in A,\ (b_1\ b_2\ \cdots\ b_s) \in B \right\}.$$

We call two-rowed arrays in $\mathscr{F}(A,B)$ F-partitions. Let $\phi_{A,B}(n)$ be the number of F-partitions in $\mathscr{F}(A,B)$, where the sum of parts in the top and bottom rows plus the number of parts in the top row equals n, and let $\bar{\phi}_{A,B}(n)$ be the number of F-partitions counted by $\phi_{A,B}(n)$ whose top and bottom rows have the same number of parts. Let $\mathscr{D}$ be the set of all partitions into distinct nonnegative parts. Then, obviously $\bar{\phi}_{\mathscr{D},\mathscr{D}}(n)$ is equal to $p(n)$, the number of partitions of n.

Let $\mathscr{O}$ be the set of all overpartitions where some parts may be 0. Then the generating function for F-partitions counted by $\phi_{\mathscr{O},\mathscr{O}}(n)$ is

$$\sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \phi_{\mathscr{O},\mathscr{O}}(n;k) z^k q^n = \frac{(-zq;q)_\infty(-z^{-1};q)_\infty}{(zq;q)_\infty(z^{-1};q)_\infty}, \tag{2.1}$$

where $\phi_{\mathscr{O},\mathscr{O}}(n;k)$ is the number of partitions counted by $\phi_{\mathscr{O},\mathscr{O}}(n)$ such that k is equal to the number of parts in the top row minus the number of parts in the bottom row.

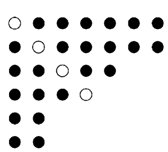

Fig. 1. Ferrers graph of $7+7+5+4+2+2$.

The right-hand side of (2.1) is related to Ramanujan's ${}_1\psi_1$ summation formula, which is studied in Section 3.

On the other hand, the constant term of (2.1), which is the generating function for F-partitions counted by $\bar{\phi}_{\mathcal{O},\mathcal{O}}(n)$, can be written as

$$\sum_{n=0}^{\infty} \bar{\phi}_{\mathcal{O},\mathcal{O}}(n)q^n = 1 + \sum_{n=1}^{\infty} \frac{(1+1)^2(1+q)^2\cdots(1+q^{n-1})^2}{(q;q)_n^2} q^n = \frac{(-q;q)_\infty^2}{(q;q)_\infty^2}, \qquad (2.2)$$

where the first equality is obtained by applying Theorem 2.1, which is proved in this section and the last equality is obtained by applying the q-Gauss summation. We show the last equality of (2.2) by proving the q-Gauss summation in a combinatorial way in Section 4.

Let $\bar{p}(n)$ be the number of overpartitions of n for which the generating function is given by (1.1), which is proved in the following theorem in a combinatorial way.

Theorem 2.1. *Let $\bar{p}(n;k)$ be the number of overpartitions of n into positive parts with k ordinary parts. Then*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \bar{p}(n;k)a^k q^n = 1 + \sum_{m=1}^{\infty} \frac{(a+1)(a+q)\cdots(a+q^{m-1})}{(q;q)_m} q^m. \qquad (2.3)$$

Proof. We prove the theorem by showing that each term on the right-hand side of (2.3) generates overpartitions into exactly m positive parts. First, we see that $q^m/(q;q)_m$ generates ordinary partitions $\lambda = (\lambda_1 \cdots \lambda_m)$ into exactly m positive parts. On the other hand, let $b = (b_1, \ldots, b_m)$ be a bit string of length m. We define $\mu = (\mu_1 \cdots \mu_m)$ by

$$\mu_i = \begin{cases} \lambda_i & \text{if } b_i = 0, \\ \overline{\lambda_i + m - i} & \text{if } b_i = 1. \end{cases}$$

We obtain an overpartition $\tilde{\mu}$ into exactly m parts rearranging μ_i in weakly decreasing order.

This algorithm is invertible. For an overpartition v, we subtract $m-j$ from v_j if v_j is overlined. Let the resulting sequence be $c = (c_1, \ldots, c_m)$. If $c_j < c_{j+1}$, we switch v_j and v_{j+1}. Successively, we apply the process above to the v obtained in the previous step. Let σ be the resulting sequence. We produce a pair of a partition $\tilde{\sigma}$ and a bit string d, where

$$\tilde{\sigma}_i = \begin{cases} \sigma_i & \text{if } \sigma_i \text{ is unrestricted}, \\ \sigma_i - m + i & \text{if } \sigma_i \text{ is overlined}, \end{cases}$$

$$d_i = \begin{cases} 0 & \text{if } \sigma_i \text{ is unrestricted}, \\ 1 & \text{if } \sigma_i \text{ is overlined}. \end{cases}$$

The process described above is the reverse of rearranging ordinary parts and overlined parts in weakly decreasing order. $\square$

Recall that $\mathcal{O}$ be the set of overpartitions into nonnegative integers. Similarly, we see that the generating function for overpartitions in $\mathcal{O}$ can be written as

$$1+\sum_{m=1}^{\infty}\frac{(a+1)(a+q)\cdots(a+q^{m-1})}{(q;q)_m}. \tag{2.4}$$

Throughout this paper, for an overpartition α in $\mathcal{O}$, we use the notation $\{\alpha\}$ and $[\alpha]$ for the array generated by $(a+1)\cdots(a+q^{m-1})$ and the partition generated by $1/(q;q)_m$, respectively, in (2.4), we make the parts of $\{\alpha\}$ corresponding to q^i in $(a+1)\cdots(a+q^{m-1})$ overlined, and we say the parts α_i of the overpartition α are in overlined order when α_i is equal to $\{\alpha\}_i+[\alpha]_i$, where $\{\alpha\}_i$ and $[\alpha]_i$ are the ith entries of $\{\alpha\}$ and $[\alpha]$, respectively.

3. Ramanujan's ${}_1\psi_1$ summation

To interpret Ramanujan's ${}_1\psi_1$ summation in a combinatorial way, we need to make some changes of variables in (1.2). Substituting azq, $-a^{-1}$, and $-bq$ for z, a, and b, respectively, we obtain

$$\begin{aligned}\frac{(-zq;q)_\infty(-z^{-1};q)_\infty}{(azq;q)_\infty(bz^{-1};q)_\infty} &= \frac{(-aq;q)_\infty}{(q;q)_\infty(abq;q)_\infty}\sum_{k=0}^{\infty}(-1/a;q)_k(-bq^{k+1};q)_\infty a^k z^k q^k \\ &\quad+\frac{(-bq;q)_\infty}{(q;q)_\infty(abq;q)_\infty}\sum_{k=-1}^{-\infty}(-1/b;q)_{-k}(-aq^{-k+1};q)_\infty b^{-k}z^k. \end{aligned}\tag{3.1}$$

When $a=b=1$, the left-hand side of (3.1) is the generating function for F-partitions counted by $\phi_{\mathcal{O},\mathcal{O}}(n)$ as seen in (2.1). On the other hand, for any k, each term on the right-hand side of (3.1) can be interpreted as the generating function for quintuples: when $k\geqslant 0$,

- two ordinary partitions,
- one partition into distinct parts,
- one partition into distinct parts greater than k,
- one overpartition into k positive parts, where the unrestricted parts are all 1's, and the overlined parts are positive and less than or equal to k,

and when $k<0$,

- two ordinary partitions,
- one partition into distinct parts,
- one partition into distinct parts greater than k,
- one overpartition into k nonnegative parts, where the unrestricted parts are all 0's, and the overlined parts are nonnegative and less than k.

Let $S(k)$ be the set of such quintuples for any integer k. Recall that $\mathscr{F}(\mathcal{O},\mathcal{O})$ is the set of F-partitions counted by $\phi_{\mathcal{O},\mathcal{O}}$. We combinatorially interpret (3.1) in the next theorem.

Theorem 3.1. *There is a one-to-one correspondence between* $\mathscr{F}(\mathcal{O},\mathcal{O})$ *and* $\bigcup_{k=-\infty}^{\infty} S(k)$ *such that a F-partition with the difference of the numbers of the top row and bottom row equal to* k *corresponds to a quintuple in* $S(k)$.

Proof. Throughout this proof, we denote the number of entries in an array c by $\ell(c)$. We rearrange parts of a given F-partition and then split the F-partition into five arrays. Note that the exponents of a and b in (3.1) indicate the numbers of unrestricted parts in the top row and the bottom row, respectively. Let a F-partition in $\mathscr{F}(\mathcal{O},\mathcal{O})$ be given by

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{r+k} \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}.$$

We consider two cases: when $k \geqslant 0$ and $k < 0$.

Case I: $k \geqslant 0$.

1. Rearrange the parts in the top row in overlined order and denote the resulting array by

$$\alpha = (\alpha_1 \ \ \alpha_2 \ \ \cdots \ \ \alpha_{r+k}).$$

Rearrange the parts in the bottom row such that the overlined parts follow the unrestricted parts, the overlined parts are in decreasing order, and the unrestricted parts are in weakly increasing order. We denote the bottom row so obtained by

$$\beta = (\beta_1 \ \ \beta_2 \ \ \cdots \ \ \beta_r).$$

2. Let μ be the array obtained by increasing each part of $\{\alpha\}$ by 1, and let λ be $[\alpha]$. We count the number of the unrestricted parts of $\{\alpha\}$ by the exponent of a, which preserves the exponent of a. We produce two partitions into distinct parts σ, v, and two ordinary partitions γ, δ as follows: for $1 \leqslant i \leqslant r$,
   - put $\beta_i + \mu_{r-i+1}$ as a part of σ if β_i is overlined and μ_{r-i+1} is unrestricted,
   - put $\beta_i + \mu_{r-i+1}$ as a part of v if β_i is unrestricted and μ_{r-i+1} is overlined,
   - put $\beta_i + \mu_{r-i+1}$ as a part of γ if β_i and μ_{r-i+1} are overlined,
   - put $\beta_i + \mu_{r-i+1}$ as a part of δ if β_i and μ_{r-i+1} are unrestricted.

   Note that the overlined parts of β and μ are distinct. Thus, σ is a partition into distinct parts the number of which are counted by a; v is the other partition into distinct parts the number of which are counted by b. Moreover, the parts of v are greater than k since μ_{r-i+1} equals $(k+i)$ if it is overlined. Note that the parts of γ are greater than or equal to $(r+k)$ since β_i and μ_{r-i+1} are greater than or equal to $(r-i)$ and $(k+i)$, respectively, if they are overlined. The number of parts of δ is counted by ab because a and b count the numbers of the unrestricted parts of μ

and β, respectively. Rearrange the parts of each partition in weakly decreasing order.

3. The remaining parts $\mu_{r+1}, \dots, \mu_{r+k}$ of μ form an overpartition ρ whose unrestricted parts are 1's, and overlined parts are positive and less than or equal to k. Note that the number of the unrestricted 1's of ρ is counted by a because a counts the number of the unrestricted parts of μ.
4. Add the parts of the conjugate of λ to γ as parts. Note that conjugate of λ has parts less than or equal to $(r+k)$, since λ has at most $(r+k)$ positive parts.

We easily see that the quintuple $(\gamma, \delta, \sigma, v, \rho)$ is in $S(k)$ and the exponents of a and b are preserved as noted in Steps 2 and 3.

The process above is reversible. Step 1 is obviously reversible. In Step 2, we make the parts of v overlined and combine v and δ to form an overpartition. As we saw in Theorem 2.1, we can locate the right places for the overlined parts, which results in arranging the parts of the overpartition in overlined order. The place for each overlined part is uniquely determined since the resulting parts have to be weakly decreasing after subtracting $(k + \ell(v) + \ell(\delta) - j)$ from the jth part if it is overlined. Similarly, we make the parts of γ overlined and combine σ and γ. We determine the places for the parts of γ using the fact that the resulting parts have to be strictly decreasing after subtracting $(k + \ell(v) + \ell(\delta) + j - 1)$ from the jth part if it is overlined. Thus Step 2 is reversible. Step 3 is obviously reversible. In Step 4, recall that $r = \ell(\sigma) + \ell(v) + \ell(\gamma) + \ell(\delta)$. Since the parts of the conjugate of λ are less than or equal to $(r+k)$ and the parts of γ are greater than or equal to $(r+k)$, the parts of the conjugate of λ are attached to the right of the rectangle of size $(r+k) \times \ell(\gamma)$. Meanwhile, if an ordinary partition is given, then there is a largest rectangle of size $(k + \ell(\sigma) + \ell(v) + \ell(\delta) + s) \times s$ for some s, which will be equal to $\ell(\gamma)$. Thus Step 4 is reversible. Since every step of the process above is reversible, so is the entire process.

We describe the reverse process. Let a quintuple $(\gamma, \delta, \sigma, v, \rho)$ in $S(k)$ be given. Recall that $\ell(c)$ denotes the number of entries in an array c. Note that the parts of a partition are in weakly decreasing order.

1. Make the parts of v overlined and decrease each part of v by $(k+1)$. Decrease each part of δ by 1. Build an array β consisting of the parts of δ and v such that parts in β are in overlined order. We use the inverse algorithm in the proof of Theorem 2.1. Increase each overlined part of $\{\beta\}$ by $(k+1)$ and add $\ell(\delta)$ 1's to $\{\beta\}$ as parts.
2. Make the parts of γ overlined. Build an array α consisting of the parts of σ and γ as follows. Let γ be $\alpha^{(0)}$. We successively build arrays $\alpha^{(i)}$ for $1 \leqslant i \leqslant \ell(\sigma)$ in the following way. For brevity, we delete the superscripts (i). For $1 \leqslant j \leqslant \ell(\alpha)$, do the following:
   - if α_j is overlined and $(\sigma_i + k + \ell(\beta) + j) \geqslant \alpha_j$, then insert σ_i before α_j; stop;
   - otherwise, increase j by 1.

   We denote the very last resulting array by α.

3. Split α into two arrays $\hat{\alpha}$ and $\hat{\mu}$ as follows. For $1\leqslant i\leqslant \ell(\alpha)$,

$$\hat{\alpha}_i = \begin{cases} \overline{\alpha_i - 1} & \text{if } \alpha_i \text{ is unrestricted,} \\ \overline{\alpha_i - (i+k+\ell(\beta))} & \text{if } \alpha_i \text{ is overlined and } \bar{\alpha}_i - (i+k+\ell(\beta)) \geqslant 0, \end{cases}$$

$$\hat{\mu}_i = \begin{cases} 1 & \text{if } \alpha_i \text{ is unrestricted,} \\ \overline{i+k+\ell(\beta)} & \text{if } \alpha_i \text{ is overlined and } \bar{\alpha}_i - (i+k+\ell(\beta)) \geqslant 0. \end{cases}$$

   We delete from α the parts corresponding to the parts of $\hat{\alpha}$ and $\hat{\mu}$.
4. Write the parts of $\hat{\mu}$ in reverse order, and attach the parts before the first part of $\{\beta\}$ and the parts of ρ after the first part of $\{\beta\}$ keeping the order of the parts of $\hat{\mu}$ and ρ. We use μ for the resulting array.
5. Take the conjugate of the partition consisting of the remaining parts of α after deleting the bars. We denote the resulting partition by λ.
6. Decrease each part of μ by 1. Add each part of μ and λ to obtain an overpartition $\tilde{\alpha}$.
7. Add the parts of $\hat{\alpha}$ as parts to $[\beta]$ to obtain an overpartition $\tilde{\beta}$.
8. Combine $\tilde{\alpha}$ and $\tilde{\beta}$ to form a F-partition.

Case II: $k<0$. By symmetry we can prove this case as we did with Case I. The only difference is that we should increase only the first $(r+k)$ parts of $\{\beta\}$ by 1. We omit the proof. $\square$

Example 1. In this example, we show how to obtain two ordinary partitions γ, δ, two partitions into distinct parts σ, v, and an array ρ using the bijection, and then show the reverse algorithm. Let

$$\begin{pmatrix} \bar{6} & \bar{\bar{5}} & \bar{4} & 4 & 3 & 2 & \bar{0} \\ 6 & \bar{\bar{7}} & 3 & \bar{3} & \bar{1} & & \end{pmatrix}$$

be a F-partition. We rearrange the top row and bottom row as in Step 1 as follows:

$$\begin{pmatrix} \bar{6} & \bar{\bar{5}} & \bar{4} & 4 & 3 & 2 & \bar{0} \\ 6 & \bar{\bar{7}} & 3 & \bar{3} & \bar{1} & & \end{pmatrix} \to \begin{pmatrix} 4 & 3 & \bar{6} & \bar{\bar{5}} & \bar{4} & 2 & \bar{0} \\ 3 & 6 & \bar{\bar{7}} & \bar{3} & \bar{1} & & \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Then,

$$\alpha = (4\ 3\ \bar{6}\ \bar{\bar{5}}\ \bar{4}\ 2\ \bar{0}) \to [\alpha] = (4\ 3\ 2\ 2\ 2\ 2\ 0), \quad \{\alpha\} = (0\ 0\ \bar{4}\ \bar{\bar{3}}\ \bar{2}\ 0\ \bar{0}).$$

Increase each part of $\{\alpha\}$ by 1 and denote the resulting array by μ. The last two parts of μ form an array ρ. Rearrange the remaining parts of μ in reverse order, and add each part to the corresponding part of β,

$$\begin{pmatrix} \bar{3} & \bar{4} & \bar{\bar{5}} & 1 & 1 \\ 3 & 6 & \bar{\bar{7}} & \bar{3} & \bar{1} \end{pmatrix}.$$

$$\bar{6} \quad \overline{10} \quad \overline{\overline{12}} \quad \bar{4} \quad \bar{2}$$

Adding 12 to the conjugate of $[\alpha]$ as a part, we obtain γ. Observe that $\bar{6}$ and $\overline{10}$ form a partition v, and $\bar{4}, \bar{2}$ form a partition σ. Let ε denote the partition with no parts,

$$\gamma = (12\ 6\ 6\ 2\ 1), \quad \underset{\substack{\uparrow \\ ab}}{\delta = \varepsilon}, \quad \underset{\substack{\uparrow \\ a}}{\sigma = (\bar{4}\ \bar{2})}, \quad \underset{\substack{\uparrow \\ b}}{v = (\overline{10}\ \bar{6})}, \quad \rho = (1\ \bar{1}).$$

In Fig. 2, the dots to the right of the rightmost vertical line form $[\alpha]$, the dots above the diagonal form $\{\alpha\}$, and the dots below the diagonal form β in the bottom row of the given F-partition. The overlined parts of $\{\alpha\}$ are marked by $*$ on the dots on the diagonal.

We apply the reverse algorithm to $(\gamma, \delta, \sigma, v, \rho)$, where $\gamma = (12\,6\,6\,2\,1), \delta = \varepsilon,\ \sigma = (4, 2),\ v = (10, 6)$ and $\rho = (1, \bar{1})$. Note that $k = 2$ since ρ has two parts. To make β, we follow Step 1 with v and δ. Since $\delta = \varepsilon$, β equals v, where $\{\beta\} = (\bar{4}, \bar{3})$ and $[\beta] = (6, 3)$. We take σ and γ, and apply Step 2 to make α. Let $\alpha^{(0)}$ be γ. We insert the parts of σ as follows. Since

$$\sigma_1 + k + l(\beta) + 1 = 4 + 2 + 2 + 1 < \alpha_1^{(0)} = \overline{12},$$

$$\sigma_1 + k + l(\beta) + 2 = 4 + 2 + 2 + 2 > \alpha_2^{(0)} = \bar{6},$$

insert σ_1 between $\alpha_1^{(0)}$ and $\alpha_2^{(0)}$. Let the resulting partition $(\overline{12}\,4\,\bar{6}\,\bar{6}\,\bar{2}\,\bar{1})$ be $\alpha^{(1)}$. Since

$$\sigma_2 + k + l(\beta) + 1 = 2 + 2 + 2 + 1 < \alpha_1^{(1)} = \overline{12},$$

$$\sigma_2 + k + l(\beta) + 3 = 2 + 2 + 2 + 3 > \alpha_3^{(1)} = \bar{6},$$

insert σ_2 between $\alpha_2^{(1)}$ and $\alpha_3^{(1)}$. Since no parts of σ remain, we stop. Let α be $(\overline{12}\,4\,2\,\bar{6}\,\bar{6}\,\bar{2}\,\bar{1})$. By Step 3, we obtain $\hat{\alpha} = (\bar{7}\,\bar{3}\,\bar{1})$ and $\hat{\mu} = (\bar{5}\,1\,1)$, α becoming $(\bar{6}\,\bar{6}\,\bar{2}\,\bar{1})$. By Step 4, we obtain $\mu = (1\,1\,\bar{5}\,\bar{4}\,\bar{3}\,1\,\bar{1})$. By Step 5, we obtain $\lambda = (4\,3\,2\,2\,2\,2)$. By

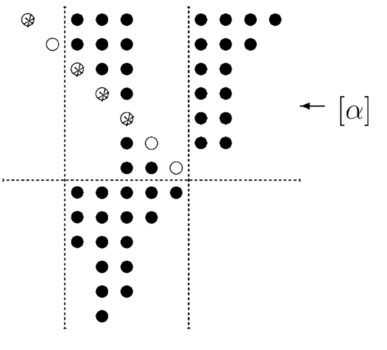

Fig. 2.

Step 6, we obtain $\tilde{\alpha} = (4\,3\,\bar{6}\,\bar{5}\,\bar{4}\,2\,\bar{0})$. By Step 7, we obtain $\tilde{\beta} = (6\,\bar{7}\,3\,\bar{3}\,\bar{1})$. By combining $\tilde{\alpha}$ and $\tilde{\beta}$, we obtain

$$\begin{pmatrix} \bar{6} & \bar{5} & \bar{4} & 4 & 3 & 2 & \bar{0} \\ 6 & \bar{7} & 3 & \bar{3} & \bar{1} & & \end{pmatrix}.$$

4. q-Gauss summation

Replacing a, b, and c by a^{-1}, b^{-1}, and cq, respectively, in the left-hand side of (1.3), we obtain

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} & \frac{(a+1)(a+q)\cdots(a+q^{n-1})(b+1)(b+q)\cdots(b+q^{n-1})}{(q;q)_n(cq;q)_n}(cq)^n \\ & = \frac{(-acq;q)_\infty(-bcq;q)_\infty}{(cq;q)_\infty(abcq;q)_\infty} \end{aligned} \tag{4.1}$$

which coincides with (2.2) when $a = b = c = 1$. Thus the left-hand side of (4.1) is the generating function for the F-partitions counted by $\bar{\phi}_{\mathcal{O},\mathcal{O}}(n)$, where the exponents of a and b are the number of unrestricted parts in the top row and the number of unrestricted parts in the bottom row, respectively. Therefore, (4.1) is a refinement of the constant term of Ramanujan's ${}_1\psi_1$ summation (3.1). In this section, we explain how the bijection established in Section 3 can be applied to q-Gauss summation.

Let a F-partition counted by $\bar{\phi}_{\mathcal{O},\mathcal{O}}(n)$ be given by

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}.$$

Rearranging the parts in the top row and dividing the resulting partition into λ and μ, we see that the exponent of c in (4.1) equals $(r + \lambda_1)$. Since we add β_i and μ_{r-i+1}, and take the conjugate of λ, we obtain exactly $(r + \lambda_1)$ parts in total, which shows that the statistic tracked by c has been preserved under the bijection.

Example 2. Let

$$\begin{pmatrix} 4 & \bar{3} & 3 & 2 & \bar{0} \\ \bar{7} & 6 & \bar{3} & 3 & \bar{1} \end{pmatrix}$$

be a F-partition. As we did in Example 1, we rearrange the top and bottom row, and then divide the top row into $[\alpha]$ and $\{\alpha\}$ as follows:

$$\begin{pmatrix} 4 & \bar{3} & 3 & 2 & \bar{0} \\ \bar{7} & 6 & \bar{3} & 3 & \bar{1} \end{pmatrix} \to \begin{pmatrix} 4 & 3 & 2 & \bar{3} & \bar{0} \\ 3 & 6 & \bar{7} & \bar{3} & \bar{1} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

$$(4\ 3\ 2\ \bar{3}\ \bar{0}) \to [\alpha] = (4\ 3\ 2\ 2\ 0), \quad \{\alpha\} = (0\ 0\ 0\ \bar{1}\ \bar{0}).$$

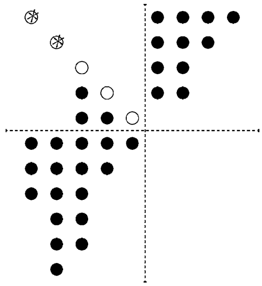

Fig. 3.

After increasing each part of $\{\alpha\}$ by 1 and rearranging them in reverse order, we add each part to the corresponding part of β:

$$\begin{array}{ccccc} \bar{1} & \bar{2} & 1 & 1 & 1 \\ 3 & 6 & \bar{7} & \bar{3} & \bar{1} \end{array}$$

$$\left(\begin{array}{ccccc} \bar{1} & \bar{2} & 1 & 1 & 1 \\ 3 & 6 & \bar{7} & \bar{3} & \bar{1} \end{array}\right).$$
$$\begin{array}{ccccc} \bar{4} & \bar{8} & \bar{8} & \bar{4} & \bar{2} \end{array}$$

We obtain the following four partitions:

$$\begin{array}{cccc} (4\ 4\ 2\ 1) & \varepsilon & (\bar{8}\ \bar{4}\ \bar{2}) & (\bar{8}\ \bar{4}) \\ \uparrow & \uparrow & \uparrow & \uparrow \\ c & abc & ac & bc \end{array}$$

Fig. 3 shows the graphical representation of Example 2.

5. Another proof of Ramanujan's ${}_1\psi_1$ summation

Corteel and Lovejoy [9] have established a combinatorial proof of the special case of Ramanujan's ${}_1\psi_1$ summation when $a = 0$. In this section, we generalize their proof to establish Ramanujan's ${}_1\psi_1$ summation for any a.

Let a F-partition counted by $\phi_{\mathcal{O},\mathcal{O}}(n)$ be given by

$$\left(\begin{array}{cccc} a_1 & a_2 & \cdots & a_{r+k} \\ b_1 & b_2 & \cdots & b_r \end{array}\right).$$

We consider two cases: $k \geqslant 0$ and $k < 0$.

Case I: $k \geqslant 0$.

1. Rearrange the bottom row in the order that overlined parts follow unrestricted parts.
2. Shift the overlined parts in the bottom row to the right so that the last part is under a_{r+k}.

3. Divide the F-partition into two arrays;
   ○ F-partition with all the overlined parts in the bottom row and corresponding parts in the top row; call it $A = \begin{pmatrix} c_1 & \cdots & c_t \\ d_1 & \cdots & d_t \end{pmatrix}$.
   ○ F-partition with the remainders; call it $B = \begin{pmatrix} \gamma_1 & \cdots & \gamma_u \\ \delta_1 & \cdots & \delta_v \end{pmatrix}$.

4. Apply the following rule to A. We denote the pair of empty partitions $(\varepsilon, \varepsilon)$ by $(\alpha^{(0)}, \beta^{(0)})$ and successively build pairs of partitions $(\alpha^{(i)}, \beta^{(i)})$ for $1 \leqslant i \leqslant t$ in the following way. For brevity, we delete the superscripts (i);
   ○ if c_{t-i+1} is overlined, then add d_{t-i+1} as a part to the conjugate of α, take the conjugate of the resulting partition, and add $(c_{t-i+1}+1)$ as a part to the partition,
   ○ if c_{t-i+1} is unrestricted, then add c_{t-i+1} as a part to α and add $(d_{t-i+1}+1)$ as a part to β.

   Let the very last pair of partitions $(\alpha^{(t)}, \beta^{(t)})$ be (α, β). Note that β is a partition into distinct parts the number of which is counted by a, since the c_i's are unrestricted and the d_i's are overlined, and α is a partition into parts less than or equal to the smallest part γ_u in the top row of B, since the c_i's are less than γ_u if overlined.
5. Rearrange the parts in the top row of B in overlined order. We denote the resulting overpartition by σ. Let μ be the array obtained by increasing each part of $\{\sigma\}$ by 1.
6. Produce a partition v into distinct parts and an ordinary partition π as follows: for $1 \leqslant i \leqslant v$,
   ○ put $\delta_i + \mu_i$ as a part of v if μ_i is overlined,
   ○ put $\delta_i + \mu_i$ as a part of π if μ_i is unrestricted.

   Note that the overlined parts of μ are distinct and in decreasing order. Thus, v is a partition into distinct parts the number of which is counted by b. The parts of v are greater than $(u-v)$, which equals k since μ_v is $(u-v+1)$ if it is overlined. We denote the partition consisting of the remaining parts of v by ρ. Note that ρ is a k-tuple of 1's and overlined integers less than or equal to k.
7. Add the parts of $[\sigma]$ to the partition α obtained in Step 4. Note that the parts of $[\sigma]$ are greater than or equal to the parts of α.
8. We obtain a quintuple $(\alpha, \beta, v, \pi, \rho)$, where α and π are ordinary partitions, β, v are two partitions into distinct parts, and ρ is an array of 1's or overlined parts less than or equal to k.

The process applied to A is the process established by Corteel and Lovejoy [9], and the process applied to B is a special case of the bijection discussed in Section 3. Thus the process above is reversible.

Case II: $k<0$. By symmetry we can proceed in this case as we did with Case I. Because the proof is similar to the previous proof, we omit it.

Example 3. Let

$$\begin{pmatrix} \bar{7} & 6 & 5 & 5 & \bar{4} & 4 & \bar{3} & 3 & \bar{1} \\ 6 & \bar{5} & 5 & \bar{4} & 4 & 3 & \bar{1} & & \end{pmatrix}$$

be an F-partition. Rearrange the parts of the bottom row such that the overlined parts follow the unrestricted parts, and divide into two arrays A and B:

$$\begin{pmatrix} \bar{7} & 6 & 5 & 5 & \bar{4} & 4 & \bar{3} & 3 & \bar{1} \\ 6 & \bar{5} & 5 & \bar{4} & 4 & 3 & \bar{1} & & \end{pmatrix} \rightarrow B = \begin{pmatrix} \bar{7} & 6 & 5 & 5 & \bar{4} & 4 \\ 6 & 5 & 4 & 3 & & \end{pmatrix},$$
$$A = \begin{pmatrix} \bar{3} & 3 & \bar{1} \\ \bar{5} & \bar{4} & \bar{1} \end{pmatrix}.$$

We build two partitions α and β by inserting each column of A:

$$\begin{array}{ccc}
\begin{pmatrix} \bar{3} & 3 & \bar{1} \\ \bar{5} & \bar{4} & \bar{1} \end{pmatrix} & \varepsilon & \varepsilon \\[2ex]
\begin{pmatrix} \bar{3} & 3 \\ \bar{5} & \bar{4} \end{pmatrix} & \begin{array}{l} \circ \\ \bullet \end{array} & \varepsilon \\[2ex]
\begin{pmatrix} \bar{3} \\ \bar{5} \end{pmatrix} & \begin{array}{ll} \bullet & \bullet \\ \circ & \\ \bullet & \end{array} & (\bar{5}) \\[2ex]
\varepsilon & \begin{array}{lll} \circ & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \circ & \\ \bullet & \bullet & \\ \bullet & & \\ \bullet & & \end{array} & (\bar{5}) \\[2ex]
 & \alpha & \beta
\end{array}$$

We divide the top row of B into λ and μ. Take the last two parts of μ for ρ and rearrange the remaining parts of μ in reverse order:

$$(\bar{7}\ 6\ 5\ 5\ \bar{4}\ 4) \rightarrow \mu = (1\ 1\ 1\ \bar{3}\ 1\ \bar{1}), \quad \lambda = (5\ 4\ 4\ 4\ 3\ 3),$$

$$\begin{array}{c} \begin{pmatrix} 1 & 1 & 1 & \bar{3} \\ 6 & 5 & 4 & 3 \end{pmatrix}, \\ \begin{array}{cccc} 7 & 6 & 5 & \bar{6} \end{array} \end{array} \quad \rho = (1\ \bar{1}).$$

Fig. 4 shows the graphical representation of Example 3, where the dots to the right of the vertical dotted line and below the horizontal dotted line represent α, the dots to the right of the vertical line and above the horizontal line represent λ, the dots to the left of the vertical line and above the horizontal line represent the parts of μ from the

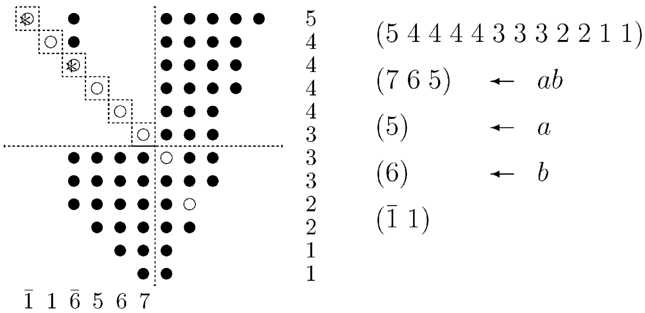

Fig. 4.

right to left, and the dots to the left of the vertical line and below the horizontal line represent the unrestricted parts of the F-partition. The rows to the right of the vertical dotted line form the partition (5 4 4 4 4 3 3 3 2 2 1 1), and the columns to the left of the vertical dotted line form the ordinary partition (7 6 5), two partitions into distinct parts (5), (6), and the overpartition ($\bar{1}$ 1).

Acknowledgments

The author is grateful to anonymous referees for their very helpful comments. The author thanks B.C. Berndt and D. Knuth for their encouragement and suggestions.

References

[1] C. Adiga, B.C. Berndt, S. Bhargava, G.N. Watson, Chapter 16 of Ramanujan's second notebook: theta-functions and q-series, Mem. Amer. Math. Soc. 53 (315) (1985).
[2] G.E. Andrews, On Ramanujan's ${}_1\psi_1(a;b;z)$, Proc. Amer. Math. Soc. 22 (1969) 552–553.
[3] G.E. Andrews, On a transformation of bilateral series with applications, Proc. Amer. Math. Soc. 25 (1970) 554–558.
[4] G.E. Andrews, Generalized Frobenius partitions, Mem. Amer. Math. Soc. 49 (301) (1984).
[5] G.E. Andrews, R. Askey, A simple proof of Ramanujan's ${}_1\psi_1$, Aequationes Math. 18 (1981) 97–100.
[6] R. Askey, Ramanujan's extensions of the gamma and beta functions, Amer. Math. Monthly 87 (1980) 346–359.
[7] W.N. Bailey, On the basic bilateral hypergeometric series ${}_2\psi_2$, Quart. J. Math. Oxford Ser. (2) 1 (2) (1950) 194–198.
[8] S. Corteel, Particle seas and basic hypergeometric series, Adv. Appl. Math. 31 (2003) 199–214.
[9] S. Corteel, J. Lovejoy, Frobenius partitions and the combinatorics of Ramanujan's ${}_1\psi_1$ summation, J. Combin. Theory Ser. A 97 (2002) 177–183.
[10] S. Corteel, J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc., to appear.
[11] N.J. Fine, Basic Hypergeometric Series and Applications, American Mathematical Society, Providence, RI, 1988.
[12] G. Frobenius, Über die Charaktere der symmetrischen Gruppe, Sitzber. Preuss. Akad., Berlin, 1900, 516–534.
[13] G. Gasper, M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.

[14] W. Hahn, Beiträge zur Theorie der Heineschen Reihen, Math. Nachr. 2 (1949) 340–379.
[15] G.H. Hardy, Ramanujan, reprinted, American Mathematical Society, Chelsea, Providence, RI, 1999.
[16] E. Heine, Untersuchungen über die Reihe

$$1+\frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)(1-q^{\gamma})}x+\frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q)(1-q^{2})(1-q^{\gamma})(1-q^{\gamma+1})}x^{2}+,\ldots$$

J. Reine Angew. Math. 34 (1847) 285–328.
[17] M.E.H. Ismail, A simple proof of Ramanujan's ${}_1\psi_1$ sum, Proc. Amer. Math. Soc. 63 (1977) 185–186.
[18] M. Jackson, On Lerch's transcendent and the basic bilateral hypergeometric series ${}_2\Psi_2$, J. London Math. Soc. 25 (1950) 189–196.
[19] J.T. Joichi, D. Stanton, Bijective proofs of basic hypergeometric series identities, Pacific J. Math. 127 (1987) 103–120.
[20] K. Mimachi, A proof of Ramanujan's identity by use of loop integral, SIAM J. Math. Anal. 19 (1988) 1490–1493.
[21] I. Pak, Partition bijections, Ramanujan J., to appear.
[22] E.M. Wright, An enumerative proof of an identity of Jacobi, J. London Math. Soc. 40 (1965) 55–57.
[23] A.J. Yee, Combinatorial proofs of generating function identities for F-partitions, J. Combin. Theory Ser. A 102 (2003) 217–228.